

INVARIANT CONVEX SETS IN POLAR REPRESENTATIONS

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ABSTRACT. We study a compact invariant convex set E in a polar representation of a compact Lie group. Polar representations are given by the adjoint action of K on \mathfrak{p} , where K is a maximal compact subgroup of a real semisimple Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subalgebra, then $P = E \cap \mathfrak{a}$ is a convex set in \mathfrak{a} . We prove that up to conjugacy the face structure of E is completely determined by that of P and that a face of E is exposed if and only if the corresponding face of P is exposed. We apply these results to the convex hull of the image of a restricted momentum map.

The boundary of a compact convex set is the union of its faces. Among the faces, the simplest ones are the exposed ones. They are given by the intersection of the convex set with a supporting hyperplane. In [3, 4] we studied the convex hull $\widehat{\mathcal{O}}$ of a K -orbit \mathcal{O} in \mathfrak{p} , where \mathfrak{p} is given by the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a reductive Lie algebra \mathfrak{g} and K acts on \mathfrak{p} by the adjoint representation. In this paper we use the results of [4] and show that a substantial part of them holds for any K -invariant compact convex set E of \mathfrak{p} . More precisely we study the faces of E . We show in Proposition 1.2 that for a face F of E there exists a subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that F is a subset of $\mathfrak{p}^{\mathfrak{s}} = \{x \in \mathfrak{p} : [x, \mathfrak{s}] = 0\}$ and F is invariant with respect to the action of $K^{\mathfrak{s}} = \{h \in K : \text{Ad}(h)(\mathfrak{s}) = \mathfrak{s}\}$, where Ad denotes the adjoint representation.

If we fix a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, then the set $P = E \cap \mathfrak{a}$ is convex and invariant with respect to the action of the normalizer $\mathcal{N}_K(\mathfrak{a}) = \{h \in K : \text{Ad}(h)(\mathfrak{a}) = \mathfrak{a}\}$ of \mathfrak{a} in K . The $\mathcal{N}_K(\mathfrak{a})$ -action on P induces an action on the set of faces of P . Similarly K acts on the set of faces of E . Denote these sets by $\mathcal{F}(P)$ respectively by $\mathcal{F}(E)$. If σ is a face of P , let σ^{\perp} denote the orthogonal complement in \mathfrak{a} of the affine hull of σ (see Section 1). Our main result is

Theorem 0.1. *The map $\mathcal{F}(P) \rightarrow \mathcal{F}(E)$, $\sigma \mapsto K^{\sigma^{\perp}} \cdot \sigma$ is well-defined and induces a bijection between $\mathcal{F}(P)/\mathcal{N}_K(\mathfrak{a})$ and $\mathcal{F}(E)/K$.*

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An application of Theorem 0.1 is the following result.

Theorem 0.2. *The faces of E are exposed if and only if the faces of P are exposed.*

Interesting K -invariant compact subsets of \mathfrak{p} often arise as images of restricted momentum or gradient mappings. More precisely, let U be a compact connected Lie group which acts by biholomorphism and in a Hamiltonian fashion on a compact Kähler manifold Z with momentum map $\mu : Z \rightarrow \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a connected Lie subgroup of $U^{\mathbb{C}}$ which is *compatible* with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that G is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. Let $X \subset Z$ be a G -invariant compact subset of Z . We have the restricted momentum map or the gradient map $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$ in the sense of [13] (see also Section 3) and we denote by $E = \widehat{\mu_{\mathfrak{p}}(X)}$ the convex hull of the K -invariant set $\mu_{\mathfrak{p}}(X)$. If \mathfrak{a} is a maximal abelian subalgebra of \mathfrak{p} and π is the orthogonal projection onto \mathfrak{a} , then $\mu_{\mathfrak{a}} = \pi \circ \mu_{\mathfrak{p}} : X \rightarrow \mathfrak{a}$ is the gradient map with respect to $A = \exp(\mathfrak{a})$. Since $P = E \cap \mathfrak{a} = \widehat{\mu_{\mathfrak{a}}(X)}$ is a convex polytope (Proposition 3.1), we deduce the following.

Theorem 0.3. *All faces of $\widehat{\mu_{\mathfrak{p}}(X)}$ are exposed.*

A reformulation of Theorem 3.1 is that the faces of E correspond to maxima of components of the gradient map. This observation will be used to realize a close connection between the faces of E and parabolic subgroups of G . More precisely, for any face $F \subset E$ let $X_F := \mu_{\mathfrak{p}}^{-1}(F)$ and let $Q^F = \{g \in G : g \cdot X_F = X_F\}$. Then X_F is the set of maximum points of an appropriately chosen component of the gradient map and Q^F is a parabolic subgroup of G .

If X is a G -stable compact submanifold of Z , then for any face F , one can construct an open neighbourhood X_F^- of X_F in X , which is an analogue of an open Bruhat cell. Moreover there is a smooth deformation retraction of X_F^- onto X_F . See Theorem 3.1 for more details.

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1. GROUP THEORETICAL DESCRIPTION OF THE FACES

We start by recalling the basic definitions and results regarding convex bodies. For more details see e.g. [18]. Let V be a real vector space with scalar product $\langle \cdot, \cdot \rangle$. A *convex body* $E \subset V$ is a convex compact subset of V . Let $\text{Aff}(E)$ denote the affine span of E . The interior of E in $\text{Aff}(E)$ is called the *relative interior* of E and is denoted by $\text{relint } E$. By definition a *face* of E is a convex subset $F \subset E$ such that $x, y \in E$ and $\text{relint}[x, y] \cap F \neq \emptyset$

implies $[x, y] \subset F$. A face distinct from E and \emptyset is called a *proper face*. The *extreme points* of E are the points $x \in E$ such that $\{x\}$ is a face. We will denote by $\text{ext } E$ the set of the extreme points of E . The set $\text{ext } E$ completely determines the convex body E since the convex hull of $\text{ext } E$ coincides with E and it is the smallest subset of E with this property. If F is a face of E , we denote by $\text{Dir}(F)$ the vector subspace of V defined by $\text{Aff}(F)$, i.e. $\text{Aff}(F) = p + \text{Dir}(F)$. We call $\text{Dir}(F)$ the *direction* of F . Every vector $\beta \in V$ defines an *exposed face* $F = F_\beta(E) = \{x \in E : \langle x, \beta \rangle = \max_{y \in E} \langle y, \beta \rangle\}$ with $\text{Dir}(F_\beta(E)) \subset \{\beta\}^\perp$. In general not all faces of a convex set are exposed, see Fig. 1 for an example. For any exposed face F the set

$$C_F = \{\beta \in V : F = F_\beta(E)\}, \quad (1)$$

is a convex cone. The faces of E are closed. If F_1 and F_2 are faces of E and they are distinct, then $\text{relint } F_1 \cap \text{relint } F_2 = \emptyset$. Moreover the convex body E is the disjoint union of the relative interiors of its faces (see [18, p. 62]).

We are interested in invariant convex bodies in polar representations. A theorem of Dadok [6] asserts that we can restrict ourselves to the following setting.

Let \mathfrak{g} be a semisimple Lie algebra with a Cartan involution θ and let B be the Killing form of \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is the eigenspace decomposition of \mathfrak{g} in 1 and -1 eigenspaces of θ and they are orthogonal under B . Moreover, B restricted to \mathfrak{k} , respectively \mathfrak{p} , is negative definite, respectively positive definite. In the sequel we denote $\langle \cdot, \cdot \rangle = B|_{\mathfrak{p} \times \mathfrak{p}}$ which is a K -invariant scalar product. Our object of study will be a K -stable convex body $E \subset \mathfrak{p}$. For for any $A, B \subset \mathfrak{p}$ we set

$$\begin{aligned} A^B &:= \{\eta \in A : [\eta, \xi] = 0, \text{ for all } \xi \in B\} \\ G^B &:= \{g \in G : \text{Ad } g(\xi) = \xi, \text{ for all } \xi \in B\}, \\ K^B &:= K \cap G^B. \end{aligned}$$

where Ad denotes the adjoint representation. In the sequel we denote by $k \cdot x = \text{Ad}(k)(x)$ the action of K on \mathfrak{p} by linear isometries.

Faces of K -invariant convex bodies in \mathfrak{p} are closely connected to orbits of subgroups of K which are given as centralizers. More precisely for any nonzero β in \mathfrak{p} we have the Cartan decomposition $\mathfrak{g}^\beta = \mathfrak{k}^\beta \oplus \mathfrak{p}^\beta$ of the Lie algebra of the centralizer G^β of β in G .

Proposition 1.1. *Let $F = F_\beta(E)$ be an exposed face of E . Then*

- a) $F \subset \mathfrak{p}^\beta$ and F is K^β -stable;
- b) $\text{Dir}(F) \subset \beta^\perp$, where \perp is in \mathfrak{p} .

Proof. If $x \in F_\beta(E)$, then $\widehat{K \cdot x} \subset E$ since E is K -invariant. Moreover, we have

$$\max_{y \in E} \langle y, \beta \rangle = \max_{y \in \widehat{K \cdot x}} \langle y, \beta \rangle = \langle x, \beta \rangle.$$

Corollary 3.1 in [4] implies $F_\beta(\widehat{K \cdot x}) \subset \mathfrak{p}^\beta$. Therefore $x \in \mathfrak{p}^\beta$. This proves a). Part b) follows since F is contained in an affine hyperplane orthogonal to β . \square

For an arbitrary face of E we have the following.

Proposition 1.2. *Let $F \subset E$ be a face. Then there exists an abelian subalgebra $\mathfrak{s} \subset \mathfrak{p}$ such that*

- a) $F \subset \mathfrak{p}^\mathfrak{s}$ and F is $K^\mathfrak{s}$ -stable;
- b) $\text{Dir}(F) \subset \mathfrak{s}^\perp$;

Proof. We may fix a maximal chain of faces $F = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k = E$ (see [3, Lemma 2]). If $k = 0$, then $F = E$ and $\mathfrak{s} = \{0\}$. Assume the theorem is true for a face contained in a maximal chain of length k . Then the claim is true for F_1 and consequently there exists $\mathfrak{s}_1 \subset \mathfrak{p}$ such that $F_1 \subset \mathfrak{p}^{\mathfrak{s}_1}$, F_1 is $K^{\mathfrak{s}_1}$ -stable and $\text{Dir}(F_1) \subset \mathfrak{s}_1^\perp$. F is an exposed face of F_1 . Let $\beta' \in \mathfrak{p}^{\mathfrak{s}_1}$ such that $F = F_{\beta'}(F_1)$ and set $\mathfrak{s} := \mathbb{R}\beta' \oplus \mathfrak{s}_1$. Then $F \subset \mathfrak{p}^\mathfrak{s}$, F is $(K^{\mathfrak{s}_1})^{\beta'} = K^\mathfrak{s}$ -stable and $\text{Dir}(F) \subset \mathfrak{s}^\perp$. \square

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of \mathfrak{p} and let $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$ be the orthogonal projection onto \mathfrak{a} . Then $P = E \cap \mathfrak{a}$ is a convex subset of \mathfrak{a} which is $\mathcal{N}_K(\mathfrak{a})$ -stable. The proof of the following Lemma is given in [7].

Lemma 1.1. *(i) If $E \subset \mathfrak{p}$ is a K -invariant convex subset, then $E \cap \mathfrak{a} = \pi(E)$ and $K \cdot \pi(E) = E$. (ii) If $C \subset \mathfrak{a}$ is a $\mathcal{N}_K(\mathfrak{a})$ -invariant convex subset, then $K \cdot C$ is convex and $\pi(K \cdot C) = C$.*

Lemma 1.2. *Let U be a compact Lie group and let $\mathfrak{g} \subset \mathfrak{u}^\mathbb{C}$ be a semisimple θ -invariant subalgebra. Then any Lie subgroup with finitely many connected components and with Lie algebra \mathfrak{g} is closed and compatible.*

Proof. We fix an embedding $U \hookrightarrow U(n)$ such that the Cartan involution $X \mapsto (X^{-1})^*$ of $\text{GL}(n, \mathbb{C})$ restricts to θ . Then G is closed in $\text{GL}(n, \mathbb{C})$ (see [16, p. 440] for a proof) and hence also in $U^\mathbb{C}$. Since \mathfrak{g} is θ -invariant, also G is, and θ restricts to the Cartan involution of G . This shows that G is compatible. \square

If $G \subset U^\mathbb{C}$ is compatible with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then \mathfrak{g} is real reductive and there is a nondegenerate K -invariant bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ which is positive definite on \mathfrak{p} , negative definite on \mathfrak{k} and such that $B(\mathfrak{k}, \mathfrak{p}) = 0$. Indeed, fix a U -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{u} . Let $\langle \cdot, \cdot \rangle$ denote also the inner product on $i\mathfrak{u}$ such that multiplication by i be an isometry of \mathfrak{u} onto $i\mathfrak{u}$. Define B on $\mathfrak{u}^\mathbb{C}$ imposing $B(\mathfrak{u}, i\mathfrak{u}) = 0$, $B = -\langle \cdot, \cdot \rangle$ on \mathfrak{u} and $B = \langle \cdot, \cdot \rangle$ on $i\mathfrak{u}$. Therefore B is $\text{Ad } U^\mathbb{C}$ -invariant and non-degenerate and its restriction to \mathfrak{g} satisfies the above conditions.

Let \mathfrak{q} be a K -invariant subspace of \mathfrak{p} . Then $[\mathfrak{q}, \mathfrak{q}]$ is a K -invariant linear subspace of \mathfrak{k} and therefore an ideal of \mathfrak{k} . Since K is compact, we have the

following K -invariant splitting $\mathfrak{k} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{k}'$. In particular \mathfrak{k}' is an ideal of \mathfrak{k} commuting with $[\mathfrak{q}, \mathfrak{q}]$. Let $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ be a K -invariant splitting of \mathfrak{p} . Since

$$B([\mathfrak{q}, \mathfrak{q}'], \mathfrak{k}) = B(\mathfrak{q}, [\mathfrak{k}, \mathfrak{q}']) \subset B(\mathfrak{q}, \mathfrak{q}') = 0,$$

this shows that $[\mathfrak{q}, \mathfrak{q}'] = 0$ and so $[\mathfrak{q}', [\mathfrak{q}, \mathfrak{q}]] = [\mathfrak{q}, [\mathfrak{q}, \mathfrak{q}']] = 0$. Moreover $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$ implies that $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{q}'$ are compatible K -invariant commuting ideal of \mathfrak{g} .

If a K -invariant linear subspace $\mathfrak{q} \subset \mathfrak{p}$ is fixed, one gets decomposition of \mathfrak{g} , and so of G . This decomposition is the content of the next Proposition. We will need it in the case where $F \subset \mathfrak{p}$ is a K -invariant convex body and \mathfrak{q} is such that $\text{Aff}(F) = x_0 + \mathfrak{q}$.

Proposition 1.3. *Let $G \subset U^{\mathbb{C}}$ be a compatible subgroup with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $\mathfrak{q} \subset \mathfrak{p}$ be a linear K -invariant subspace. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$ where $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$ and $\mathfrak{h}' = \mathfrak{h}^{\perp B}$. Then the following hold.*

- a) \mathfrak{h} and \mathfrak{h}' are compatible K -invariant commuting ideal of \mathfrak{g} ;
- b) Let K_1 be the connected Lie subgroup of G with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}, \mathfrak{h}]$. Then $K_1 \exp(\mathfrak{q})$ is a connected compatible subgroup of G and any two maximal subalgebras of \mathfrak{q} are conjugate by an element of K_1 .
- c) Let K_2 be the connected Lie subgroup of G with Lie algebra $\mathfrak{k} \cap [\mathfrak{h}', \mathfrak{h}']$. Then any two maximal subalgebras of \mathfrak{q}' are conjugate by an element of K_2 .

Proof. We have proved (a) in the above discussion. Let $\mathfrak{b} := [\mathfrak{h}, \mathfrak{h}]$. Then $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{b}$ and \mathfrak{b} is semisimple. Denote by B the connected subgroup of $U^{\mathbb{C}}$ with Lie algebra \mathfrak{b} . By Lemma 1.2 B is a closed subgroup of $U^{\mathbb{C}}$. Set $\mathfrak{z}_{\mathfrak{p}} := \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{p}$ and $\mathfrak{d} := \mathfrak{b} \oplus \mathfrak{a}$. Then \mathfrak{d} is a reductive Lie algebra and $\exp \mathfrak{a}$ is a compatible abelian subgroup commuting with B . Thus $D := B \cdot \exp \mathfrak{a}$ is a connected closed subgroup with Lie algebra \mathfrak{d} . Moreover $D \cap U = B \cap U$ and $\exp(\mathfrak{b} \cap \mathfrak{p}) \cdot \exp \mathfrak{a} = \exp(\mathfrak{b} \cap \mathfrak{p} \oplus \mathfrak{a}) = \exp(\mathfrak{d} \cap \mathfrak{p})$. This shows that D is compatible. Since $D \cap U$ coincides with K_1 and D is connected the last statement in (b) follows from standard properties of compatible subgroups (see e.g. Prop. 7.29 in [16]; note that a connected compatible subgroup is a reductive group in the sense of [16, p. 446]). This proves (b). For (c) the same argument applies more directly. It is enough to observe that the connected Lie subgroup $H'' \subset G$ with Lie algebra $[\mathfrak{h}', \mathfrak{h}']$ is semisimple, compatible and connected and that $K_2 = H'' \cap U$. \square

Remark 1.1. *The compatible subgroup G in the previous Proposition is not assumed to be connected. Nevertheless the constructions in (b) and (c) depend only on G^0 . Thus considering G^0 in place of G makes no difference.*

Lemma 1.3. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a reductive Lie algebra and \mathfrak{g}_i ideals. If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal subalgebra, then $\mathfrak{a}_i := \mathfrak{a} \cap \mathfrak{p}_i$ is a maximal subalgebra of \mathfrak{p}_i and $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$.*

If σ is a face of P , let σ^{\perp} denote the orthogonal (inside \mathfrak{a}) to the direction of the affine hull of σ .

Lemma 1.4. *Let F be a face and let \mathfrak{s} be as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing \mathfrak{s} . Set $\sigma := \pi(F)$. Then σ is a face of P , $\sigma = F \cap \mathfrak{a}$ and $F = K^{\sigma^\perp} \cdot \sigma$. Moreover F is a proper face if and only if $F \cap \mathfrak{a}$ is.*

Proof. By Proposition 1.2 $F \subset \mathfrak{p}^{\mathfrak{s}}$ is a $K^{\mathfrak{s}}$ -stable convex set. By Lemma 1.1 we get $\sigma = \pi(F) = F \cap \mathfrak{a}$ and this is a face P by [3, Lemma 11]. Since $\text{Dir}(F)$ is contained in the orthogonal complement of \mathfrak{s} , and $\text{Dir}(\sigma) \subset \text{Dir}(F)$, we have $\text{Dir}(\sigma) \subset \mathfrak{a} \cap \mathfrak{s}^\perp$. Then $\sigma^\perp \subset \mathfrak{s}$. Hence $K^{\sigma^\perp} \cdot \sigma \subset K^{\mathfrak{s}} \cdot \sigma \subset F$. We prove the reverse inclusion. If $y \in F$, then $F \cap \widehat{K \cdot y}$ is a face of $\widehat{K \cdot y}$. Set $\tilde{\sigma} = \pi(F \cap \widehat{K \cdot y})$. We have $\tilde{\sigma} \subset \sigma$ and by Proposition 3.6 in [4] we also have that $F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma}$. On the other hand, $\sigma^\perp \subset \tilde{\sigma}^\perp$, so $K^{\tilde{\sigma}^\perp} \subset K^{\sigma^\perp}$ and

$$F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma} \subset K^{\sigma^\perp} \cdot \sigma.$$

This implies $F = K^{\sigma^\perp} \cdot \sigma$. Note that F is proper if σ is. It remains to prove that σ is proper, when F is proper.

Let $\text{Aff}(E) = x_o + \mathfrak{q}_E$. Note that $\mathfrak{q}_E = \{x - y : x, y \in \text{Aff}(E)\}$ implies that \mathfrak{q}_E is K -invariant. Since K acts on \mathfrak{p} by isometries, we may assume that x_o is orthogonal to \mathfrak{q} . Note that x_o is uniquely defined by this condition. It follows that x_o is a K fixed point and $E = x_o + E_1$, where E_1 is a K -invariant convex body of \mathfrak{q}_E . Proposition 1.3 applied to \mathfrak{q}_E yields K_1, K_2 such that $G_1 = K_1 \exp(\mathfrak{q}_E)$ is a connected compatible semisimple real Lie group, $K = K_1 \cdot K_2$ and for any $x \in E$ we have

$$K \cdot x = K \cdot (x_o + x_1) = x_o + K \cdot x_1 = x_o + K_1 \cdot x_1 = K_1 \cdot x.$$

since \mathfrak{q}_E is fixed pointwise by K_2 . By Lemma 1.3, $\mathfrak{a} = \mathfrak{a}_E \oplus \mathfrak{a}'_E$, where \mathfrak{a}_E is a maximal abelian subalgebra of \mathfrak{q}_E and \mathfrak{a}'_E is a maximal abelian subalgebra of \mathfrak{q}'_E . Since $\pi(E) = \pi(x_o) + \pi(E_1)$ and $\text{Dir}(E_1) = \mathfrak{q}_E$, it follows that the direction of $\pi(E)$ is \mathfrak{a}_E . If $\sigma = \pi(F) = \pi(E) = E \cap \mathfrak{a}$, then $\sigma^\perp = \mathfrak{a}'_E$ and so $K_1 \subset K^{\mathfrak{a}'_E}$. It follows that

$$F = K^{\mathfrak{a}'_E} \cdot (E \cap \mathfrak{a}) = K_1 \cdot (E \cap \mathfrak{a}) = K \cdot (E \cap \mathfrak{a}) = E.$$

where the last equality follows by Lemma 1.1. Hence, if F is proper, then $\sigma = \pi(F) \subsetneq \pi(E) = E \cap \mathfrak{a}$. \square

Proposition 1.4. *Let F be a proper face and let \mathfrak{s} as in Proposition 1.2. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra containing \mathfrak{s} . Then F is exposed if and only if $F \cap \mathfrak{a}$ is.*

Proof. Assume that there exists $\beta \in \mathfrak{p}$ such that $F = F_\beta(E)$. Since $F \cap \mathfrak{a} = \sigma$ is a proper face of P , the point β is not orthogonal to \mathfrak{a} . We have $\beta = \beta_1 \oplus \beta_2$, with $\beta_1 \in \mathfrak{a}$ different from zero and β_2 orthogonal to \mathfrak{a} . Therefore $F_\beta(E) \cap \mathfrak{a} = F_{\beta_1}(E) \cap \mathfrak{a} = F_{\beta_1}(P) = \sigma$. Now, assume that there exists $\beta \in \mathfrak{a}$ such that $\sigma = F_\beta(P)$. Let $F' := F_\beta(E)$. By Proposition 1.1 $F' \subset \mathfrak{p}^\beta$. Moreover $\mathfrak{a} \subset \mathfrak{p}^\beta$ since $\beta \in \mathfrak{a}$. By Lemma 1.4 the intersection of a face with

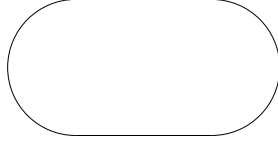


FIGURE 1.

\mathfrak{a} determines the face. Since $F' \cap \mathfrak{a} = F_\beta(P) = \sigma = F \cap \mathfrak{a}$ we conclude that $F = F'$. Thus F is exposed. \square

Remark 1.2. *Given a Weyl-invariant convex body $P \subset \mathfrak{a}$, set $E := K \cdot P$. By Lemma 1.1 E is a K -invariant convex body in \mathfrak{p} and $P = E \cap \mathfrak{a}$. Thus a general P can be realized as $E \cap \mathfrak{a}$. A general Weyl-invariant convex body P can have non-exposed faces. For example take $G = U^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ and $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$. Then $\mathfrak{a} = \mathbb{R}^2$ and the Weyl group is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ where the generators are given by the reflections on the axes. The picture in Fig. 1 is a Weyl-invariant P with exactly 4 non-exposed faces. By the Proposition also the corresponding body $E \subset \mathfrak{isu}(2) \oplus \mathfrak{isu}(2)$ has non-exposed faces.*

2. PROOF OF THE MAIN RESULTS

Let $\mathfrak{a} \subset \mathfrak{p}$ and define the following map

$$\Upsilon : \mathcal{F}(P) \longrightarrow \mathcal{F}(E), \quad \sigma \mapsto K^{\sigma^\perp} \cdot \sigma$$

Since σ is $\mathcal{N}_{K^{\sigma^\perp}}(\mathfrak{a})$ -invariant, it follows from Lemma 1.1 that $\Upsilon(\sigma)$ is a face of E .

Theorem 0.1. *The map Υ induces a bijection between $\mathcal{F}(P)/\mathcal{N}_K(\mathfrak{a})$ and $\mathcal{F}(E)/K$.*

Proof. Set $\mathcal{N} := \mathcal{N}_K(\mathfrak{a})$. We first show that Υ is \mathcal{N} -equivariant. Let $w \in \mathcal{N}$. Then $\sigma' = w\sigma$ implies $K^{\sigma'^\perp} = wK^{\sigma^\perp}w^{-1}$ and therefore $\Upsilon(\sigma') = w\Upsilon(\sigma)$. This means that the map

$$\tilde{\Upsilon} : \mathcal{F}(P)/\mathcal{N} \longrightarrow \mathcal{F}(E)/K, \quad [\sigma] \mapsto K^{\sigma^\perp} \cdot \sigma$$

is well-defined. Next, we prove that $\tilde{\Upsilon}$ is injective. Assume for some $g \in K$ $g \cdot F_1 = F_2$ where $F_1 = \Upsilon(\sigma_1)$ and $F_2 = \Upsilon(\sigma_2)$. Since $F_2 = K^{\sigma_2^\perp} \cdot \sigma_2$, the face F_2 is a $K^{\sigma_2^\perp}$ -invariant convex body. Moreover $\sigma_2 \subset \mathfrak{a} \subset \mathfrak{p}^{\sigma_2^\perp}$ and $\mathfrak{p}^{\sigma_2^\perp}$ is $K^{\sigma_2^\perp}$ -invariant. Therefore F_2 is contained in $\mathfrak{p}^{\sigma_2^\perp}$. It follows that $\mathrm{Aff}(F_2) = x_o + \mathfrak{q}_{F_2}$, where \mathfrak{q}_{F_2} is a $K^{\sigma_2^\perp}$ invariant subspace of $\mathfrak{p}^{\sigma_2^\perp}$, x_o is a fixed $K^{\sigma_2^\perp}$ point and it is orthogonal to \mathfrak{q}_{F_2} . We apply Proposition 1.3 to the group $G^{\sigma_2^\perp}$ and \mathfrak{q}_{F_2} . Thus $\mathfrak{h}_{F_2} = [\mathfrak{q}_{F_2}, \mathfrak{q}_{F_2}] \oplus \mathfrak{q}_{F_2}$ and its orthogonal complement in $\mathfrak{g}^{\sigma_2^\perp}$, that we denote by \mathfrak{h}'_{F_2} , are commuting ideal. The Proposition 1.3 also yields subgroups $K_1, K_2 \subset K^{\sigma_2^\perp}$ such that any two maximal subalgebras in \mathfrak{q}_{F_2} , respectively \mathfrak{q}'_{F_2} , are interchanged by

K_1 , respectively K_2 . Since $\sigma_2 \subset \mathfrak{a}$, also $\text{Dir}(\sigma_2) \subset \mathfrak{a}$ and we may decompose $\mathfrak{a} = \text{Dir}(\sigma_2) \oplus \sigma_2^\perp$. But $\text{Dir}(\sigma_2)$ is contained also in \mathfrak{q}_{F_2} since $\sigma_2 \subset F_2$. So $\sigma_2^\perp \subset \mathfrak{q}_{F_2}^\perp \cap \mathfrak{p} = \mathfrak{q}'_{F_2}$. By dimension $\text{Dir}(\sigma_2)$ is a maximal subalgebra in \mathfrak{q}_{F_2} and σ_2^\perp is a maximal subalgebra in \mathfrak{q}'_{F_2} . On other hand from $g \cdot F_1 = F_2$ it follows that $g \cdot \text{Dir}(\sigma_1) \subset \mathfrak{q}_{F_2}$ and $g \cdot \sigma_1^\perp \subset \mathfrak{q}'_{F_2}$, and they are also maximal subalgebras in these spaces. By the Proposition 1.3 (b) and (c) there exist $k_1 \in K_1, k_2 \in K_2$ such that

$$\begin{aligned} (k_1 g) \cdot \text{Dir}(\sigma_1) &= \text{Dir}(\sigma_2) \\ (k_2 g) \cdot \sigma_1^\perp &= \sigma_2^\perp. \end{aligned}$$

Since x_0 is fixed by the larger group $K^{\sigma_2^\perp}$ it follows that $k_1 g \sigma_1 = \sigma_2$. Moreover $k_1 k_2 = k_2 k_1$ since $[\mathfrak{h}_{F_2}, \mathfrak{h}'_{F_2}] = 0$. For the same reason \mathfrak{q}'_{F_2} is fixed pointwise by K_1 and \mathfrak{q}_{F_2} is fixed pointwise by K_2 . Set $k = k_1 k_2$ and $w = k g$. Then $k \in K^{\sigma_2^\perp}$ and $w \in K$. We get

$$\begin{aligned} w \cdot \text{Dir}(\sigma_1) &= \text{Dir}(\sigma_2) \\ w \cdot \sigma_1^\perp &= \sigma_2^\perp. \end{aligned}$$

Thus $w \cdot \mathfrak{a} = \mathfrak{a}$, i.e. $w \in \mathcal{N}$. Since $k \in K^{\sigma_2^\perp}$, $w \cdot F_1 = (k g) \cdot F_1 = k \cdot F_2 = F_2$. Since $\sigma_1 = (x_0 + \text{Dir}(\sigma_1)) \cap F_1$ and similarly for F_2 , we conclude that $w \sigma_1 = \sigma_2$. Finally we prove that $\tilde{\Theta}$ is surjective. Let $F \subset \hat{\mathcal{O}}$ be a face. Then $F \subset \mathfrak{p}^\mathfrak{s}$ for some abelian subalgebra $\mathfrak{s} \in \mathfrak{p}$. Then there exists $k \in K$ such that $k \cdot \mathfrak{a} \subset \mathfrak{p}^\mathfrak{s}$. Therefore $k^{-1} \cdot F \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$ and $\mathfrak{a} \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$. By Proposition 1.4, $k \cdot F = K^{\sigma^\perp} \cdot \sigma$ where $\sigma = (k \cdot F) \cap \mathfrak{a}$ and so $\tilde{\Upsilon}$ is surjective. \square

As an application of the above theorem and Proposition 1.4, we get the following result.

Theorem 0.2. *The faces of E are exposed if and only if the faces of P are exposed.*

Proof. By the above Theorem, the map $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$ induces a bijection between $\mathcal{F}(P)/\mathcal{N}$ and $\mathcal{F}(E)/K$. Hence, keeping in mind that if $F_1 = k F_2$, then F_1 is exposed if and only if F_2 , the result follows from Proposition 1.4. \square

Remark 2.1. *We have proven Theorems 0.1 and 0.2 under the assumption that G is a connected real semisimple Lie group. From this it follows that both theorems hold true for any connected compatible subgroup of $U^\mathbb{C}$, since such a subgroup is real reductive in the sense of [16, p. 446] and thus it is the product of a semisimple connected subgroup and an abelian subgroup, see e.g. [16, p. 453].*

3. CONVEX HULL OF THE GRADIENT MAP IMAGE

Let U be a compact connected Lie group and $U^\mathbb{C}$ its complexification. Let (Z, ω) be a Kähler manifold on which $U^\mathbb{C}$ acts holomorphically. Assume

that U acts in a Hamiltonian fashion with momentum map $\mu : Z \rightarrow \mathfrak{u}^*$. Let $G \subset U^{\mathbb{C}}$ be a closed connected subgroup of $U^{\mathbb{C}}$ which is compatible with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This means that G is a closed subgroup of $U^{\mathbb{C}}$ such that $G = K \exp(\mathfrak{p})$, where $K = U \cap G$ and $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ [13, 15]. The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction a K -equivariant map $\mu_{i\mathfrak{p}} : Z \rightarrow (i\mathfrak{p})^*$. Using a fixed U -invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{u} , we identify $\mathfrak{u} \cong \mathfrak{u}^*$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product on $i\mathfrak{u}$ such that multiplication by i be an isometry of \mathfrak{u} onto $i\mathfrak{u}$. For $z \in Z$ let $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$ denote $-i$ times the component of $\mu(z)$ in the direction of $i\mathfrak{p}$. In other words we require that $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$, for any $\beta \in \mathfrak{p}$. Then we view $\mu_{i\mathfrak{p}}$ as a map

$$\mu_{\mathfrak{p}} : Z \rightarrow \mathfrak{p},$$

which is called the G -gradient map or restricted momentum map associated to μ . For the rest of the paper we fix a G -stable compact subset $X \subset Z$ and we consider the gradient map $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$ restricted on X . We also set

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu^{-i\beta}.$$

We will now study the convex hull of $\mu_{\mathfrak{p}}(X)$, that we denote by E . Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of \mathfrak{p} and let $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$ be the orthogonal projection onto \mathfrak{a} . Then $\pi \circ \mu_{\mathfrak{p}} =: \mu_{\mathfrak{a}}$ is the gradient map associated to $A = \exp(\mathfrak{a})$. Let Z^A be the set of fixed points of A . We note that $\mu_{\mathfrak{a}}$ is locally constant on Z^A since $\text{Ker } d\mu_{\mathfrak{a}}(x) = (\mathfrak{a} \cdot x)^{\perp}$ (see [15]). Let \mathfrak{b} a subspace of \mathfrak{a} and let $X^{\mathfrak{b}} = \{p \in X : \xi_X(p) = 0 \text{ for all } \xi \in \mathfrak{b}\}$, where ξ_X is the vector field induced by the A action on X . Then the map $\mu_{\mathfrak{b}} : X^{\mathfrak{b}} \rightarrow \mathfrak{b}$, that is the composition of $\mu_{\mathfrak{p}}$ with the orthogonal projection onto \mathfrak{b} , is locally constant ([11], Section 3). Since $X^{\mathfrak{b}}$ is compact, $\mu_{\mathfrak{b}}(X^{\mathfrak{b}})$ is a finite set. In [11] it also shown that for any $y \in X^{(\mathfrak{b})} := \{p \in X : \mathfrak{a}_p = \mathfrak{b}\}$, where $\mathfrak{a}_p := \{\xi \in \mathfrak{a} : \xi_X(p) = 0\}$, we have that $\mu_{\mathfrak{a}}(A \cdot y) \subset \mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ is an open subset of the affine space $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (the orthogonal complements are taken in \mathfrak{a}). Moreover $\mu_{\mathfrak{a}}(A \cdot y)$ is a convex subset of $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$ (see [10]) and therefore $\mu_{\mathfrak{a}}(\overline{A \cdot y}) = \overline{\mu_{\mathfrak{a}}(A \cdot y)}$ is a convex body.

Let $P := \mu_{\mathfrak{a}}(X)$. If $\beta \in \mu_{\mathfrak{a}}(X)$ is an extremal point of P , and $y \in \mu_{\mathfrak{a}}^{-1}(\beta)$, then $\mu_{\mathfrak{a}}(A \cdot y)$ is an open neighborhood of $\mu_{\mathfrak{a}}(y)$ in $\mu_{\mathfrak{a}}(y) + \mathfrak{a}_y^{\perp}$ and it is contained in $\mu_{\mathfrak{a}}(X) \subset P$. Since $\mu_{\mathfrak{a}}(y)$ is an extremal point, it follows that $\mathfrak{a}_y^{\perp} = \{0\}$ and so y is a fixed point of A . Since X is compact, the set X^A has finitely many connected components. Therefore P has finitely many extremal points, i.e. it is a polytope. We have shown the following.

Proposition 3.1. *Let $X \subset Z$ be a G -invariant compact subset of Z . Then the image $\mu_{\mathfrak{a}}(X^A)$ is a finite set $\{c_1, \dots, c_p\}$ and $P = \overline{\mu_{\mathfrak{a}}(X)}$ is the convex hull of c_1, \dots, c_p .*

As a corollary we get the following result.

Theorem 0.3. *Let $X \subset Z$ be a G -invariant compact subset of Z . Then every face of $E = \widehat{\mu_{\mathfrak{p}}(X)}$ is exposed.*

Proof. Since

$$\pi(E) = \pi(\widehat{\mu_{\mathfrak{p}}(X)}) = \widehat{\mu_{\mathfrak{a}}(X)},$$

by Lemma 1.1 (i) we conclude that $E \cap \mathfrak{a} = \pi(E) = P$ and by Proposition 3.1, Remark 2.1 and Theorem 0.2 we get that every face of E is exposed. \square

We call P the *momentum polytope*. If $G = U^{\mathbb{C}}$ and X is a complex connected submanifold of Z , then $P = \mu_{\mathfrak{a}}(X)$ by the Atiyah-Guillemin-Sternberg convexity theorem [1, 8]. The same holds for X an irreducible semi-algebraic subset of a Hodge manifold Z [17, 11, 5].

Since any proper face F of E is exposed, the set C_F defined in (1) is a non-empty convex cone in \mathfrak{p} . Set

$$K^F := \{g \in K : g \cdot F = F\}.$$

By Proposition 5 in [3] we have $C_F^{K^F} := \{\beta \in C_F : K^F \cdot \beta = \beta\} \neq \emptyset$. This means that for a proper face F one can find a K^F -invariant vector β such that $F_{\beta}(E) = F$. For $\beta \in \mathfrak{p}$, denote by X^{β} the set of points of X that are fixed by $\exp(\mathbb{R}\beta)$. If $\beta \in C_F$, let

$$X_{\max}^{\beta} := \{x \in X : \mu_{\mathfrak{p}}^{\beta}(x) = \max_X \mu_{\mathfrak{p}}^{\beta}\}.$$

Since the function $\mu_{\mathfrak{p}}^{\beta}$ is K^{β} -invariant the set X_{\max}^{β} is K^{β} -invariant. Moreover X_{\max}^{β} is a union of finitely many connected components of X^{β} and X^{β} is G^{β} -stable. Every connected component of G^{β} meets K^{β} . This implies that G^{β} leaves X_{\max}^{β} invariant. Next we show that X_{\max}^{β} does not depend on the choice of β in C_F .

Lemma 3.1. *If $\beta \in C_F$, then $X_{\max}^{\beta} = \mu_{\mathfrak{p}}^{-1}(F)$. Moreover F is the convex hull of $\mu_{\mathfrak{p}}(X_{\max}^{\beta})$.*

Proof. Fix $x \in X$. Then $\mu_{\mathfrak{p}}(x) \in F$ if and only if $\langle \mu_{\mathfrak{p}}(x), \beta \rangle = \max_{v \in E} \langle v, \beta \rangle$. Moreover $\max_{v \in E} \langle v, \beta \rangle = \max_{v \in \mu_{\mathfrak{p}}(X)} \langle v, \beta \rangle = \max_X \mu_{\mathfrak{p}}^{\beta}$. So $x \in \mu_{\mathfrak{p}}^{-1}(F)$ if and only if x is a maximum of $\mu_{\mathfrak{p}}^{\beta}(x)$ restricted to X . This shows that $X_F^{\beta} = \mu_{\mathfrak{p}}^{-1}(F)$. The inclusion $\mu_{\mathfrak{p}}(X_F^{\beta}) \subset F$ follows from the definition and therefore $\widehat{\mu_{\mathfrak{p}}(X_F^{\beta})} \subset F$. By [3, Lemma 3] $\widehat{\text{ext } F} = \text{ext } E \cap F$, so $\text{ext } F \subset \widehat{\mu_{\mathfrak{p}}(X) \cap F} = \widehat{\mu_{\mathfrak{p}}(X_F^{\beta})}$. It follows that $F = \widehat{\mu_{\mathfrak{p}}(X_F^{\beta})}$. \square

Motivated by the above Lemma we set $X_F := X_{\max}^{\beta}$ where β is any vector in C_F . We also set

$$Q^F = \{g \in G : g \cdot X_F = X_F\}.$$

Q^F is a closed Lie subgroup of G .

Given $\beta \in \mathfrak{p}$ define the following subgroups:

$$\begin{aligned} G^{\beta+} &= \{g \in G : \lim_{t \rightarrow -\infty} \exp(t\beta)g \exp(-t\beta) \text{ exists}\}, \\ G^{\beta-} &= \{g \in G : \lim_{t \rightarrow +\infty} \exp(-t\beta)g \exp(t\beta) \text{ exists}\}, \\ R^{\beta+} &= \{g \in G : \lim_{t \rightarrow -\infty} \exp(t\beta)g \exp(-t\beta) = e\}, \\ R^{\beta-} &= \{g \in G : \lim_{t \rightarrow +\infty} \exp(-t\beta)g \exp(t\beta) = e\}. \end{aligned}$$

$G^{\beta+}$ (respectively $G^{\beta-}$) is a parabolic subgroup, $R^{\beta+}$ (respectively $R^{\beta-}$) is its unipotent radical and G^β is a Levi factor. Therefore $G^{\beta+} = G^\beta \rtimes R^{\beta+}$ (respectively $G^{\beta-} = G^\beta \rtimes R^{\beta-}$).

Lemma 3.2. $Q^F \cap K = K^F$.

Proof. If $g \in Q^F \cap K$, then $g \cdot X_F = X_F$. Since $\mu_{\mathfrak{p}}$ is a K -invariant map, $g \cdot \mu_{\mathfrak{p}}(X_F) = \mu_{\mathfrak{p}}(X_F)$. Taking the convex hull of both sides and using Lemma 3.1 we get that $g \cdot F = F$, thus $g \in K^F$. Conversely, if $g \in K^F$, the equivariance of $\mu_{\mathfrak{p}}$ yields $X_F = \mu_{\mathfrak{p}}^{-1}(F) = \mu_{\mathfrak{p}}^{-1}(g \cdot F) = gX_F$, thus $g \in Q^F$. \square

We are now ready to prove the connection between the set of the faces of E and parabolic subgroups of G .

Proposition 3.2. Q^F is a parabolic subgroup of G . Moreover $Q^F = G^{\beta+}$ for every $\beta \in C_F^{K^F}$.

Proof. Observe that by definition Q^F is a closed subgroup of G . Let $\beta \in C_F^{K^F}$. Then $F = F_\beta(E)$ and, by definition of K^F , we get $K^F = K^\beta$. The set $X_F = \{x \in X : \mu_{\mathfrak{p}}^\beta(x) = \max_X \mu_{\mathfrak{p}}^\beta\}$ is G^β -stable. Fix $p \in X_F$ and consider the orbit $G \cdot p$, which is a smooth submanifold contained in X . By Proposition 2.5 in [13] (see also Proposition 2.1 in [4]) we get that $\xi_X(x) = 0$ for any $\xi \in \mathfrak{r}^{\beta+}$ and for any $x \in X_F$. Therefore $G^{\beta+} \cdot p \subset X_F$. Hence $G^{\beta+} \subset Q^F$ and the Lie algebra \mathfrak{q}^F of Q^F is parabolic. On the other hand by Lemma 3.2, we have $\mathfrak{q}^F \cap \mathfrak{k} = \mathfrak{g}^{\beta+} \cap \mathfrak{k} = \mathfrak{k}^\beta$ and so by Lemma 3.7 [4] we conclude that $\mathfrak{q}^F = \mathfrak{g}^{\beta+}$. Since $Q^F \subset N_G(\mathfrak{g}^{\beta+}) = G^{\beta+}$ we get $Q^F = G^{\beta+}$. \square

Remark 3.1. If $\beta' \in C_F^{K^F}$, then $Q_F = G^{\beta'+} = G^{\beta+}$. By Lemma 2.8 in [4], we have $[\beta, \beta'] = 0$, $G^\beta = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$.

Let $Q^{F-} = \Theta(Q^F)$, where $\Theta : G \rightarrow G$ denotes the Cartan involution. The subgroup Q^{F-} is parabolic and depends only on F . The subgroup $L^F := Q^F \cap Q^{F-}$ is a Levi factor of both Q^F and Q^{F-} . Let $\beta \in C_F^{K^F}$. Then $Q^F = G^{\beta+}$, $L^F = G^\beta$ and we have the projection

$$\pi^{\beta+} : G^{\beta+} \rightarrow G^\beta, \quad \pi^{\beta+}(g) = \lim_{t \rightarrow +\infty} \exp(t\beta)g \exp(-t\beta),$$

respectively

$$\pi^{\beta+} : G^{\beta-} \longrightarrow G^{\beta}, \quad \pi^{\beta-}(g) = \lim_{t \rightarrow -\infty} \exp(t\beta)h \exp(-t\beta).$$

Lemma 3.3. *If $\beta \in C_F^{K^F}$, then the projections $\pi^{\beta+}$ and $\pi^{\beta-}$ depend only on F .*

Proof. Let $g \in G^{\beta+}$. We know that $g = hr$, where $h \in G^{\beta}$ and $r \in R^{\beta+}$. Then

$$\pi^{\beta+}(g) = \lim_{t \rightarrow +\infty} \exp(t\beta)g \exp(-t\beta) = h \lim_{t \rightarrow +\infty} \exp(t\beta)r \exp(-t\beta) = h.$$

Since $G^{\beta} = G^{\beta'}$ and $R^{\beta+} = R^{\beta'+}$ the decomposition $g = hr$ is the same for both groups and $\pi^{\beta+}(g) = \pi^{\beta'+}(g)$. The same argument works for $\pi^{\beta-}$. \square

Now assume that X is a G -stable compact submanifold of Z .

For $\beta \in C_F^{K^F}$ set $X_F^{\beta-} := \{p \in X : \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \in X_F\}$. Then the map

$$p^{\beta-} : X_F^{\beta-} \longrightarrow X_F, \quad p^{\beta-}(x) = \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot x \quad (2)$$

is well-defined, G^{β} -equivariant, surjective and its fibers are $R^{\beta-}$ -stable.

More generally one can consider $p^{\beta-}$ as a map from $X^{\beta-} = \{y \in X : \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot y \text{ exists}\}$ to X^{β} . In general however this map is not even continuous [14, Example 4.2]. To ensure continuity and smoothness it is enough that the topological Hilbert quotient $X^{\beta-}/G^{\beta}$ exists. Using the notation of [14] and choosing $r = \max_X \mu_p^{\beta}$, we have $X_F = X_{\max}^{\beta} = X_r^{\beta}$ and $X_r^{\beta-} = X_F^{\beta-}$. Thus Prop. 4.4 of [14] applies and yields that $X_F^{\beta-}$ is an open $G^{\beta-}$ -stable subset of X and that (2) is smooth deformation retraction onto X_F . Using $\pi^{\beta-}$ one defines an action of $Q^{F-} = G^{\beta-}$ on X_F by setting $g \cdot x = \pi^{\beta-}(g) \cdot x$. This just depends on F . With respect to this action the map $p^{\beta-}$ becomes Q^{F-} -equivariant.

Lemma 3.4. *The set $X_F^{\beta-}$ and the map $p^{\beta-}$ do not depend on the choice of $\beta \in C_F^{K^F}$.*

Proof. Set $\Gamma = \exp(\mathbb{R}\beta)$. If $p \in X_F$ by the Slice Theorem [13, Thm. 3.1] there are open neighborhoods $S_p \subset T_p X$ and $\Omega_p \subset X$ and a Γ -equivariant diffeomorphism $\Psi_p : S_p \longrightarrow \Omega_p$, such that $0 \in S_p$, $p \in \Omega_p$, $\Psi_p(0) = p$. Since p is a maximum of μ_p^{β} restricted to X , the following orthogonal splitting $T_p X = V_0 \oplus V_-$ with respect to the Hessian of μ_p^{β} holds. Here V_0 denotes the kernel of the Hessian of μ_p^{β} and V_- denotes the sum of eigenspaces of the Hessian of μ_p^{β} corresponding to negative eigenvalues. We also point out that $V_0 = T_p X_F$ and $S_p = \{x_0 + x_- : x_0 \in S_p \cap V_0, x_- \in V_-\}$, see [15]. It follows that $\Omega_p \subset X_F^{\beta-}$. Set $\Omega := \bigcup_{p \in X_F} \Omega_p$. By what we just proved, $\Omega \subset X_F^{\beta-}$. On the other hand Ω is an open Γ -invariant neighbourhood of X_F , so $X_F^{\beta-} \subset \Omega$. So $X_F^{\beta-} = \Omega$. If β' is another vector of $C_F^{K^F}$, set

$B = \exp(\mathbb{R}\beta \oplus \mathbb{R}\beta')$. This is a compatible abelian subgroup and $X_F \subset X^B$. So we may choose the open subsets Ω_p above to be B -stable. Therefore we get $X^{\beta'} = \Omega$ as well. This proves that $X_F^{\beta^-} = X_F^{\beta'^-}$.

Next we show that $p^{\beta^-} = p^{\beta'^-}$. First observe that $p^{\beta^-}(y) = p^{\beta'^-}(y)$ if $y \in \Omega$. Indeed if $y \in \Omega_p$ we can study the limit using the diffeomorphism $\Psi_p : S_p \rightarrow \Omega_p$. The decomposition $T_p X = V_0 \oplus V_-$ is the same for β and β' since they commute and attain their maxima on X_F . Therefore if $x = \Psi_p^{-1}(y) = x_0 + x_-$, then

$$p^{\beta^-}(y) = \Psi_p(x_0) = p^{\beta'^-}(y). \quad (3)$$

If $p \in X_F^{\beta^-}$ and $q = \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \in X_F$, there is $t_1 \in \mathbb{R}$, such that $\exp(t\beta) \cdot p \in \Omega$. Therefore

$$\begin{aligned} \lim_{t \rightarrow +\infty} \exp(t\beta') \cdot p &= \lim_{t \rightarrow +\infty} \exp(t\beta')(\exp(t_1\beta') \cdot p) \\ &= \lim_{t \rightarrow +\infty} \exp(t\beta)(\exp(t_1\beta') \cdot p) \quad (\text{by 3}) \\ &= \exp(t_1\beta') \left(\lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \right) \\ &= \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p. \end{aligned}$$

□

By the above Lemma if F is a face and $\beta \in C_F^{K^F}$, we can set $X_F^- := X_F^{\beta^-}$ and $p^{F^-} := p^{\beta^-} : X_F^- \rightarrow X_F$.

Theorem 3.1. *For any face $F \subset E$, the set X_F is closed and L^F -stable, X_F^- is an open Q^{F^-} -stable neighborhood of X_F in X and the map p^{F^-} is a smooth Q^{F^-} -equivariant deformation retraction of X_F^- onto X_F .*

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